ON THE PERIAPSIS ADVANCE OF POLAR ORBITS IN THE KERR FAMILY OF SPACETIMES

Ekaterini STOGHIANIDIS and Dimitri TSOUBELIS

Department of Physics, University of Ioannina, Ioannina, Greece

Received 20 March 1986; accepted for publication 24 April 1986

The orbits of bound particles passing through the symmetry axis of the Kerr spacetime are considered A simple expression is derived giving the periapsis advance of the almost spherical subclass of such polar orbits in terms of the mass and spin parameters of the Kerr solution of Einstein's equations.

The magnetic-like components of the gravitational field produced by rotating objects give rise to the interesting class of phenomena coming under the heading of inertial frame dragging. These effects were first studied in the context of the Thirring and Lense [1] approximate solution of Einstein's equations. It was thereby discovered that the rotation of the gravitating mass forces the line of nodes of bound non-equatorial orbits to advance in the sense in which the central mass is rotating. Actually, this effect is supposed to be measurable in the case of the gravitational field of the earth, as argued by van Patten and Everitt [2], who have proposed a method of measuring it using two satellites in polar orbit.

Since the Lense–Thirring metric agrees asymptotically with that of Kerr, one expects that all effects associated with geodesic motion in the former solution will be present in the geodesic orbits of the latter. Indeed, the extensive qualitative as well as quantitative analysis of geodesic motion in the Kerr spacetime, which has been carried out by several investigators, especially after Carter [3] proved the separability of the corresponding Hamilton–Jacobi equation, has shown that this is indeed the case. (See ref. [4] for a thorough presentation of this analysis as well as an extensive list of the relevant references.) Thus, it has been shown that, in general, bound timelike orbits are drastically affected by the angular momentum of the "source" and wander around in space over a wide range of the angular coordinates, ϑ and φ, of the Boyer–Lindquist (t, r, ϑ, φ) coordinate system.

However, even when complete integration of the geodesic equations is feasible, the resulting expressions are quite complicated. This masks the fact that the qualitative features of geodesic motion in the Kerr spacetime are essentially simple and renders the comparison with the Schwarzschild case unreasonably difficult.

In the present paper we attempt to lift this handicap. This is made possible by restricting our considerations to the subset of almost spherical timelike orbits passing through the symmetry axis of the Kerr spacetime, as such polar orbits exhibit all the interesting features of bound timelike geodesics not hitting the horizon. An expression for the azimuthal and latitudinal precession of the periapsis of such orbits is derived which, more than being simple, is valid over the whole range of values of the radial coordinate for which stable orbits are allowed, including, of course, the asymptotic Lense–Thirring region.

Consider, therefore, a particle of rest mass μ moving in the Kerr spacetime. As shown by Carter [3], its equations of motion admit four first integrals one of which reads

$$\rho^4\dot{\vartheta}^2 = Q - a^2\Gamma^2\cos^2\vartheta - L_z^2\cot^2\vartheta. \tag{1}$$

Here

$$\rho^2 = r^2 + a^2\cos^2\vartheta, \tag{2}$$

0375-9601/86/$03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

$$Q = K - (L_z - aE)^2, \tag{3}$$

$$\Gamma^2 \equiv 1 - E^2, \tag{4}$$

and a dot denotes differentiation with respect to the proper time parameter, τ. E and L_z stand for the particle's energy and angular momentum along the symmetry axis, respectively, and K is Carter's constant. For convenience, the fourth constant of the motion, μ, was set equal to unity.

It follows from (1) that in order for the orbit to reach the polar axis, where $\cos^2\vartheta = 1$, it is necessary that

$$L_z = 0. \tag{5}$$

The same equation gives

$$k^2 \equiv a^2\Gamma^2/Q < 1 \tag{6}$$

as the condition for bound orbits, corresponding to $\Gamma^2 > 0$, to reach the axis from points where $\cos^2\vartheta \neq 0$.

Eq. (5) leads to considerable simplification of the remaining equations of motion. They become

$$\rho^4\dot{r}^2 = R(r) \equiv (r^2 + a^2)^2[E^2 - V^2(r)], \tag{7}$$

$$\rho^2\dot{\varphi} = 2aMEr/\Delta(r), \tag{8}$$

and

$$\rho^2\dot{t} = -a^2E\sin^2\vartheta + E(r^2 + a^2)^2/\Delta(r), \tag{9}$$

where

$$\Delta(r) = r^2 - 2Mr + a^2 \tag{10}$$

and

$$V^2(r) = \Delta(r)(K + r^2)/(a^2 + r^2)^2. \tag{11}$$

As spherical orbits one defines the orbits for which $r = r_0$, a constant. This means that r_0 is a double root of the function $R(r)$ defined by (7). Thus, for a spherical polar orbit

$$R(r) = (r - r_0)^2G(r), \tag{12}$$

where

$$G(r) = -\Gamma^2r^2 + 2(M - \Gamma^2r_0)r - a^2Q/r_0^2. \tag{13}$$

In this case, the constants of the motion are determined by r_0. Specifically,

$$\Gamma^2 = \frac{M(r^4 - 4Mr^3 + 2a^2r^2 + a^4)}{(a^2 + r^2)(r^3 - 3Mr^2 + a^2r + Ma^2)} \tag{14}$$

and

$$Q = \frac{Mr^2(r^4 + 2a^2r^2 - 4Ma^2r + a^4)}{(r^2 + a^2)(r^3 - 3Mr^2 + a^2r + Ma^2)}, \tag{15}$$

where, for convenience, r_0 has been replaced by r. Using the last two equations and (6), one obtains

$$k^2 = \frac{(a/r)^2(r^4 - 4Mr^3 + 2a^2r^2 + a^4)}{(r^4 + 2a^2r^2 - 4Ma^2r + a^4)}. \tag{16}$$

Eqs. (1) and (7)–(9) have been integrated by Wilkins [5], for the case of spherical orbits with $a^2 = 1$ and by Johnston and Ruffini [6] for $a^2 < 1$.

When the particle coordinate r varies, (1) and (7) can be combined to give

$$\mathrm{d}r/\sqrt{R(r)} = \mathrm{d}\psi/\sqrt{Q(1 - k^2\sin^2\psi)} \tag{17}$$

where the latitude angle ψ is defined by

$$\psi = \tfrac{1}{2}\pi - \vartheta. \tag{18}$$

Thus

$$\int \mathrm{d}r/\sqrt{R(r)} = F(\psi, k)/\sqrt{Q}, \tag{19}$$

where $F(\psi, k)$ is the elliptic integral of the first kind.

Suppose, now, that the particle under consideration has energy, E, slightly different from the energy, E_0, of a stable spherical orbit at r_0, but the same K. Then, one can use (7) and (12) to write

$$R(r) = (r^2 + a^2)^2[E^2 - E_0^2 + E_0^2 - V^2(r)]$$
$$= (r^2 + a^2)^2(E^2 - E_0^2) + (r - r_0)^2G(r). \tag{20}$$

It follows from this equation that the turning points of the r-motion are located at coordinate value r_1 determined by

$$(r_1 - r_0)^2 = -(r_1^2 + a^2)^2(E^2 - E_0^2)/G(r_1), \tag{21}$$

which is close to r_0 due to the smallness of $(E^2 - E_0^2)$. This, in turn, allows one to write

$$R(r) \approx -G(r_0)\left[(r_1-r_0)^2-(r-r_0)^2\right]. \quad (22)$$

and, then,

$$\int \frac{\mathrm{d}r}{\sqrt{R(r)}} \approx \left[-G(r_0)\right]^{-1/2} \frac{\sin^{-1}(r-r_0)}{|r_1-r_0|}. \quad (23)$$

Thus, (19) yields

$$r \approx r_0 + |r_1-r_0| \sin\{m[F(\psi, k) - F(\psi_0, k)]\}, \quad (24)$$

where

$$m^2 = -G(r_0)/Q. \quad (25)$$

Eq. (25) implies that the orbiting particle completes a radial oscillation when the latitude changes by an amount $\delta\psi = \psi - \psi_0$ determined by the equation

$$F(\psi, k) - F(\psi_0, k) = 2n/m. \quad (26)$$

Correspondingly, the azimuthal coordinate increases by $\delta\varphi$, which, to the same order of approximation, is given by

$$\delta\varphi = \frac{2aMEr_0}{\Delta(r_0)Q^{1/2}} \frac{2n}{m}. \quad (27)$$

Substituting (13)–(15) into (26), one obtains the exact expression

$$m^2 = (r^6 - 6Mr^5 + 3a^2r^4 + 4Ma^2r^3 + 3a^4r^2 - 6Ma^4r + a^6) \times \left[r^2(r^4 + 2a^2r^2 - 4Ma^2r + a^4)\right]^{-1}, \quad (28)$$

giving the quantity m in terms of the mass and spin parameters, M and a, of the Kerr spacetime and the radius, r, of the associated stable spherical orbit.

As our main result, given by (26) and (27), involves elliptic integrals, it seems a bit complicated. The range of values which the parameter k takes, however, allows for a simplification. Specifically, k remains quite small even when an extreme Kerr black hole is considered, for which $a^2 = M^2$. In this case, for example, $k^2 < 0.011$ down to the last stable orbit which is located at $r \approx 5.275M$. As a result, the approximate expression

$$F(\psi_0, k) \approx \psi + (k/2)^2(\psi - \sin\psi\cos\psi) \quad (29)$$

for the elliptic integral is quite satisfactory. Thus, the change of the angle of latitude can be written in the form

$$\delta\psi = \delta\psi^{(0)} - (k/2)^2\left[\delta\psi^{(0)} - \sin\delta\psi^{(0)} \times \cos(\delta\psi^{(0)} + \psi_0)\right], \quad (30)$$

where

$$\delta\psi^{(0)} = 2n/m. \quad (31)$$

A further simplification obtains, when one is interested in the asymptotic region where the Kerr solution of Einstein's equations reduces to the Lense–Thirring one. In this region (16) and (28) become

$$k^2 = (a/r)^2 \quad (32)$$

and

$$m^2 = 1 - 6M/r + a^2/r^2, \quad (33)$$

respectively, while (30) reduces to

$$\delta\psi = \delta\psi^{(0)}\left[1 - k^2\sin^2(\psi_0/2)\right]. \quad (34)$$

Finally, when the spin parameter, a, vanishes and the Kerr metric reduces to that of Schwarzschild, (26) and (27) give the well-known result [7]

$$\delta\varphi = 0, \quad \delta\psi = 2n/\sqrt{1-6M/r}, \quad (35)$$

corresponding to motion in a fixed plane.

References

[1] H. Thirring, Phys. Z 19 (1918) 33;
J. Lense and H. Thirring, Phys. Z. 19 (1918) 156,
B. Mashhoon, F.W. Hehl and D S. Theiss, Gen Rel. Grav. 16 (1984) 711.
[2] R.A. van Patten and C.W.F Everitt, Phys Rev. Lett 36 (1976) 629.
[3] B. Carter, Phys. Rev. 174 (1968) 1559
[4] S. Chandrasekhar, The mathematical theory of black holes (Oxford Univ. Press, Oxford, 1983) ch. 7.
[5] D C. Wilkins, Phys. Rev. D 5 (1972) 814.
[6] M Johnston and R Ruffini, Phys. Rev. D 10 (1974) 2324.
[7] C.W Misner, K.S Thorne and J A. Wheeler, Gravitation (Freeman, San Francisco, 1973) p. 670